Journal of the Earth & Space Physics. Vol. 37, No. 4, 2012, P. 13-26

Linear gravity inversion including the minimum moment of inertia

Ardestani, E. V.[1*] and Nejati Kalate, A.[2]

[1] Associate Professor, Earth Physics Department, Institute of Geophysics, University of Tehran and Center of Excellence in Survey Engineering and Disaster Management, Tehran, Iran
[2] Assistant Professor, Mining, Oil and Geophysics engineering Department, Shahrood University of Technology

(Received: 6 Oct 2008 , Accepted: 8 Feb 2011)

Abstract
The compact gravity inversion including the minimization of moment of inertia has been applied to determine the geometry of anomalous bodies which cause much more depth esolution.

The new algorithm is based on Lewi's (1996) procedure including the minimum moment of inertia. The method is used with good results to several 3-Dimensional synthetic models and real examples.

The advantage of using this combination method is presented by comparing it with the other methods.

Key words: Compact gravity inversion, Moment of inertia, 3-Dimensional prisms

وارون‌سازی خطی داده‌های گرانی با استفاده از کمینه مقدار ممان اینرسی

وحید ابراهیم‌زاده اردستانی[1] و علی نجاتی کلاته[2]

[1] دانشیار، گروه فیزیک زمین، مؤسسهٔ ژئوفیزیک دانشگاه تهران و قطب علمی مهندسی نقشه‌برداری و مقابله با سوانح طبیعی، تهران، ایران
[2] استادیار، گروه مهندسی معدن، نفت و ژئوفیزیک دانشگاه صنعتی شاهرود، شاهرود، ایران

(دریافت: ۸۷/۷/۱۵ ، پذیرش نهایی: ۸۹/۱۱/۱۹)

چکیده
روش معکوس خطی فشرده داده‌های گرانی با در نظر گرفتن گشتاور لختی (moment of inertia) برای تعیین هندسه بی‌هنجاری‌های گرانی در این مقاله به کار رفته است. ترکیب روش وارون فشرده با حداقل گشتاور لختی باعث افزایش دقت مدل‌سازی با افزایش عمق می‌شود. آلگوریتم جدیدی که در این تحقیق به کار رفته نتیجه افزودن کمینه گشتاور لختی به آلگوریتم به کار رفته ازسوی لوی (۱۹۹۶) است. این آلگوریتم به‌نحو موفقیت‌آمیزی برای مدل‌های مصنوعی سه‌بُعدی و داده‌های واقعی به کار رفته است. مزیت استفاده از این روش نسبت به روش‌های دیگر نیز نشان داده شده است.

واژه‌های کلیدی: وارون سازی فشرده، ممان اینرسی، بلوک‌های سه بعدی

1 INTRODUCTION

The main target of gravity interpretation is to deduce a plausible causative subsurface body from surface observation which is the definition of gravity inversion.

The gravity inversion is nonunique. In other words, there may exist several density distributions that produce the same gravity effect at the surface that only one of them is the real one.

In order to over come the non uniqueness of gravity interpretation, inversion methods look for a solution which determine the density or geometrical properties in accordance with assumptions that are known as constraints.

Two approaches are taken. The first searches for the shape of source structures knowing their contrast as well as certain

*Corresponding author: Tel:021- 61118232 Fax: 021- 88630548 E-mail: ebrahimz@ut.ac.ir

constraints on their geometry.

The second type of approach searches for the contrast distribution in a domain knowing a partition of it into elementary cells as well as constraints on the contrast.

Using this approach Last and Kubik (1982) suggested seeking the source distribution with minimum volume to explain the anomaly and is named compact gravity inversion.

The principle used is to minimize the volume of the causative body, which is equivalent the maximizing its compactness.

In this case the relationship between the parameters and the observations is linear and the compactness criterion has been used to reduce the ambiguity of the results.

Although Last and Kubik (1983) approach leads to geologically more appropriate structures, the bodies obtained are often too expensive horizontally and in some cases remain too shallow.

The technique is broadened to include the search for the solutions minimizing the moment of inertia with respect to the center of gravity or with respect to a given dip line passing through it by Guillen and Menichetti (1984).

Barbosa, et al., (1994) generalized the methodology of Guillen et al., (1984) for compactness along several axes using Tikhonov's regularization.

Lewi (1997) has also improved the original compact inversion (Last, and Kubik, (1983)) by introducing a new approach to the 3D compact gravity inversion.

We aim to use the minimization of the moment of inertia through Lewi's algorithm.

2 MODEL

The model used here is the one with fixed geometry consisting of rectangular prisms whose densities are allowed to vary individually.

The exact expression of the gravitational attraction of a rectangular prism at an arbitrary point P in space which lies out of the causative body, is given by Banerjee and Gupta (1977),

$$g = f\,\sigma\Big[x\ln\big(y+\sqrt{x^2+y^2+z^2}\big) + y\ln\big(x+\sqrt{x^2+y^2+z^2}\big) - z\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}\Big]\Bigg|_{x1}^{x2}\Bigg|_{y1}^{y2}\Bigg|_{z1}^{z2} \qquad (1)$$

where f is universal gravitational constant and σ is the density of the prism and g is the vertical attraction of a prism that is bounded by the planes *X=x1, X=x2;*
Y=y1, Y=y2; Z=z1, Z=z2 at an arbitrary position *P* out of the prism in space.

3 METHOD

The domain in which the anomalous sources are searched is divided into elementary rectangular prisms. The elementary density contrasts are constant inside each prism and can vary individually.

With the matrix notation the gravity anomalies measured on N points (G=g_j, j=1,…,N) is given by,

$$\mathbf{G=AX+E} \qquad (2)$$

Where **A** is the contribution of the prism i with a unit density in measurement point j which could be computed through eq. (1) and **X** is the contrast density of the prism i (σ in eq. (1)) and **E** represent the noise at data points.

If one assumes that the signal and noise in eq. (2) are Gaussian random variables Then the best approximation ($\overline{X}$) to the true parameters in eq. (2) can be achieved using the well known stochastic inversion procedure.

The solution of the system of the equations in (2) in an stochastic inversion process and for an under-determined system is as follows (Tarantola, 1987)

$$\overline{X} = C_m A^T (A C_m A^T + C_e)^{-1} G \qquad (3)$$

where C_m and C_e are the parameter and the error covariance matrices respectively.

It is also usual to take the weighting matrices W_m and W_e (Koch 1988) in place of the covariance matrices and using the following definition

$$W_e = \alpha C_e^{-1} \qquad and \qquad W_m = \beta C_m^{-1} \quad (4)$$

Substituting eqn.(4) in (3) we get,

$$\overline{X} = W_m^{-1} A^T (A W_m^{-1} A^T + \frac{\alpha}{\beta} W_e^{-1})^{-1} G \quad (5)$$

where W_m and W_e are parameters and errors weighting matrices and $\frac{\alpha}{\beta}$ must be defined properly.

Last and Kubik (1983) used the density of each block as weighting matrix W_m to get a compact subsurface mass distribution in two-dimensional data inversion.

They have used the a priori noise to signal ration instead of $\frac{\alpha}{\beta}$ and defined W_m^{-1} and W_e^{-1} as following equations,

$$[W_m^{(k)}]_{jj}^{-1} = ([\overline{X}_j^2]^{(k-1)} + \eta) \qquad j = 1,...,m \quad (6)$$

$$W_e^{-1} = diag[A W_m^{-1} A^T] \quad (7)$$

Here k stands for the number of iteration, *diag* for diagonal and η is a very small constant in the order of the machine accuracy.

In their method for overcoming the non uniqueness of the solutions they defined the density constraint (X_0) obtained from a priori information.

Then any block that exceeds the density barriers will be set to X_0 and the algorithm automatically freezes this block in the next iteration by assigning a very small weight to it. So in each step one has to compute,

$$g_i^{*(k)} = g_i - X_0 \sum_j a_{ij} \Theta[\overline{X}_j^{(k-1)} / X_0] \quad (8)$$

where Θ denotes the unit Heaviside step function and $g_i^{*(k)}$ is the reduced gravity data of the k iteration at the measuring point *i*. Similarly the modification of the weighting matrices and the solution will be,

$$[W_m^{*(k)}]_{jj}^{-1} = \eta + [\overline{X}_j^{(k-1)}]^2 \{1 - \Theta[\overline{X}_j^{(k-1)} / X_0]\} \quad (9)$$

$$\mu^{(k)} = \mu^{(k-1)} \frac{\left| g_i^* - e_i \right|_{\max}^{(k-1)}}{\left| g_i^* - e_i \right|_{\max}^{(k)}} \quad (10)$$

$$\overline{X}_j^{(k)} = [D^{(k)} G^{*(k)}]_j + x_0 \Theta[\overline{X}_j^{(k-1)} / X_0] \quad (11)$$

Where

$$D^{(k)} = [W_m^{*(k)}]^{-1} A^T (A [W_m^{*(k)}]^{-1} A^T + \mu^{(k)} [W_e^{*k}])^{-1} \quad (12)$$

and $\mu^{(k)}$ which is defined as the ratio of the signal to noise is derived in each step by multiplying the former one $\mu^{(k-1)}$ to the ratio of the maximum signal amplitudes before and after removal or addition the block(s).

The working principle of this method has been tested for two-dimensional error- free data (Last and Kubik, 1983).

Some instabilities have been reported (Lewi et al. 1994) in Last and Kubik (1983) method when complicated mass distribution is used.

Therefore Lewi (1996) improved the Last and Kubik's method by defining the parameter and error weighting matrices as follows,

$$W_g = \frac{1}{(\sigma_g)^2} C_g^{-1} \qquad and \qquad W_m = \frac{1}{(\sigma_m)^2} C_m^{-1} \quad (13)$$

where σ_g and σ_m are certain variances which means that they could be any definition of variances of the data and the parameters (Ilk, 1993) and C_m and C_g are the error and data covariance matrices respectively.

Substituting equation (13) in equation (5) yields,

$$\overline{X} = W_m^{-1} A^T (A W_m^{-1} A^T + \mu \frac{\sigma_m^2}{\sigma_g^2} W_g^{-1})^{-1} G \quad (14)$$

where μ is the regularization parameter and defined by Lewi as,

$$\mu^{(k)} = \frac{\sigma_g^2}{1+(\sigma_e^2)^{(k-1)}} \quad (15)$$

By giving equal weight to data and substituting eqn.(15) in eqn. (14) we have,

$$\overline{X} = W_m^{-1} A^T \left(A W_m^{-1} A^T + \frac{\sigma_m^2 I}{1+\sigma_e^2} \right)^{-1} G \quad (16)$$

where I represents identity matrix and we have,

$$[\sigma_m^2]^{(k)} = \frac{\sum_{j=1}^{m} [\overline{X}_j]^{(k-1)}}{(m-1)}$$

and

$$[\sigma_e^2]^{(k)} = \frac{\sum_{i=1}^{n} \{g_i - \sum_{j=1}^{m} a_{ij} [\overline{X}_j]^{(k-1)}\}^2}{(n-1)} \quad (17)$$

4 MOMENTS OF INERTIA

According to Guillen and Menichetti (1984) the moment of inertia M is the sum of the individual moments,

$$M = \sum_i^N M_i \quad (18)$$

as the density is constant in each block *i*, the moment has the form

$$M_i = \Omega_i v_i (K_i^2 + d_i^2) \quad (19)$$

As Last and Kubik (1983) posed the problem in the form of weighted least squares, to find the parameters X (v_i , *i*=1, ….,M in eq. (19)), the weight is defined as,

$$M_i = W_i v_i^2 \quad (20)$$

Thus for the weight (W_i) we can write,

$$W_i = \frac{\Omega_i (K_i^2 + d_i^2)}{|v_i| + \varepsilon} \quad (21)$$

where Ω is the volume of the prism i , K_i is the coefficient depending on the form of the element i , d_i is the distance from the center of the gravity of block *i* to the total center of the gravity, v_i is the contrast density of the prism *i* and ε is chosen to be sufficiently small according to the computer.

When we are dealing with the moments of inertia about center of gravity for 3-D rectangular prisms (Guillen and Menicheetti, 1984), k_i has the form,

$$K_i^2 = \frac{a^2 + b^2 + c^2}{3} \quad (22)$$

where a,b and c are the dimensions of the prism.

5 NUMERICAL PROCEDURE

At the first stage we used Lewi's algorithm consisting of following steps,

1- determining the machine accuracy.

2- Compute the kernel A via eqn. (1).

For the first iteration an identity matrix is used in place of the parameter weighting matrix and σ_m and σ_e are set to zero. Therefore the solution of the first iteration is the least square solution.

3- Compute the new value of σ_m and σ_e through equations (17) and (18).

4- Using eqn.(9) which assigns a very small weight (η) to the prisms whose densities have crossed the target densities and compute the parameter weighting matrix for those blocks that have not crossed the target density.

5- Start the new iteration by removing the effect of those blocks that have just crossed the target density using eqn.(10).

6- Carrying out the inversion through eqn. (11) where for computing D^k eqn.(16) is applied.

7- Repeat the procedure 4 to 6 until the criteria for convergence is full filled.

The criterion is the root-mean square (RMS) of the differences between the observed and modified gravity effects which must be equal or smaller than a pre described one.

8- Plotting the results.

Then we incorporated the minimum moment of inertia in inversion process by substituting

(21) into eqn. (9).

9- For final comparison the Guillen and Menichetti (1984) method is used for inversion applying equations (8), (10), (11), (12) and (21).

6 SYNTHETIC MODELS

The first synthetic model is a rectangular prism (Fig1). The gravity effect of this prism plus 10 percent of the gravity effect as the noise is the input data for the inversion process.

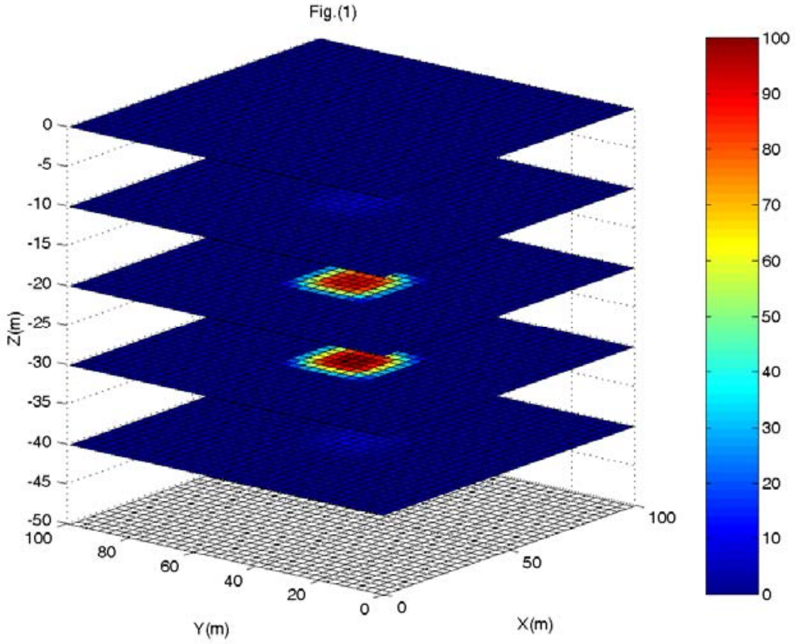

Figure 1. Synthetic model.

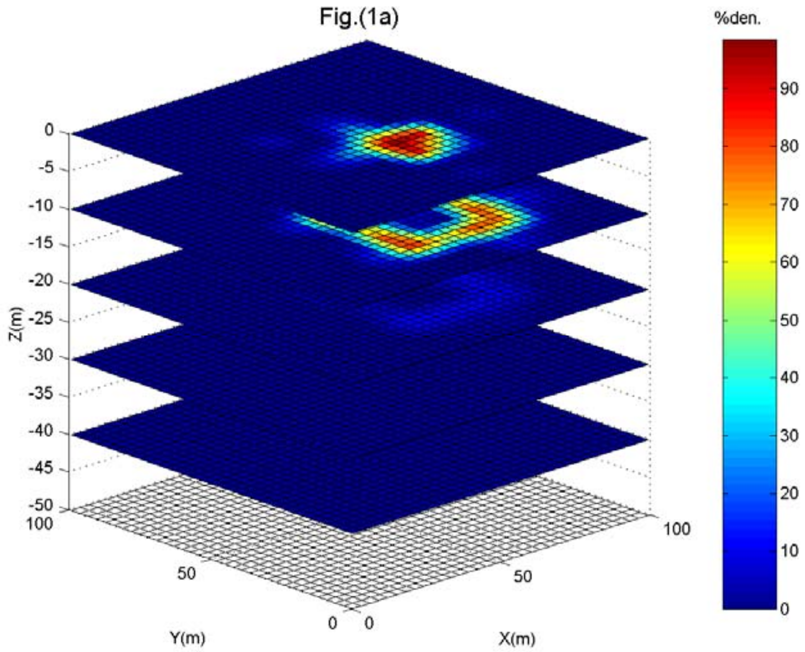

Figure 1(a). Inversion result by Lewi's algorithm.

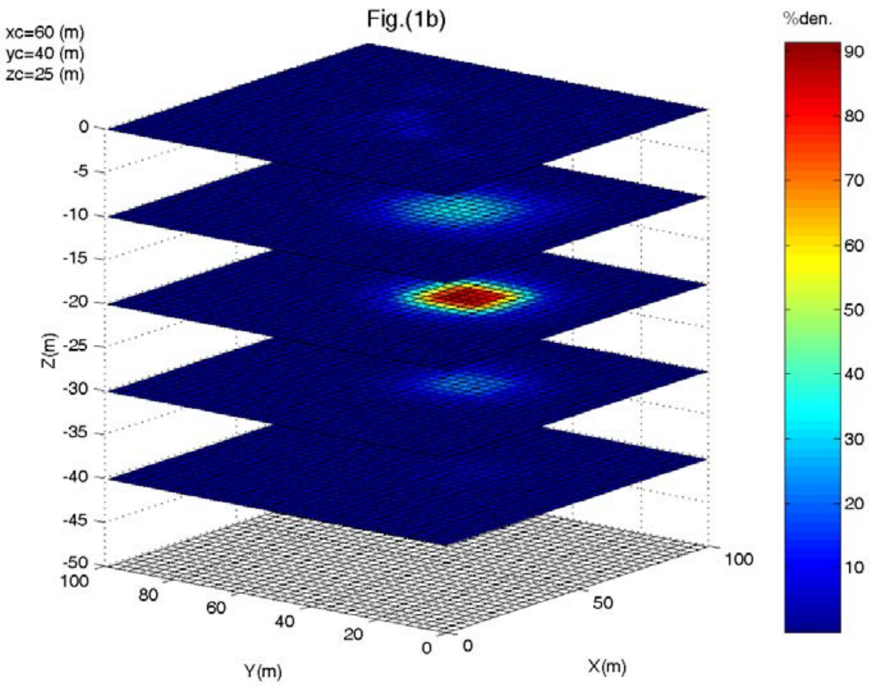

Figure 1(b). Inversion results by Lewi's algorithm including the minimum moment of inertia.

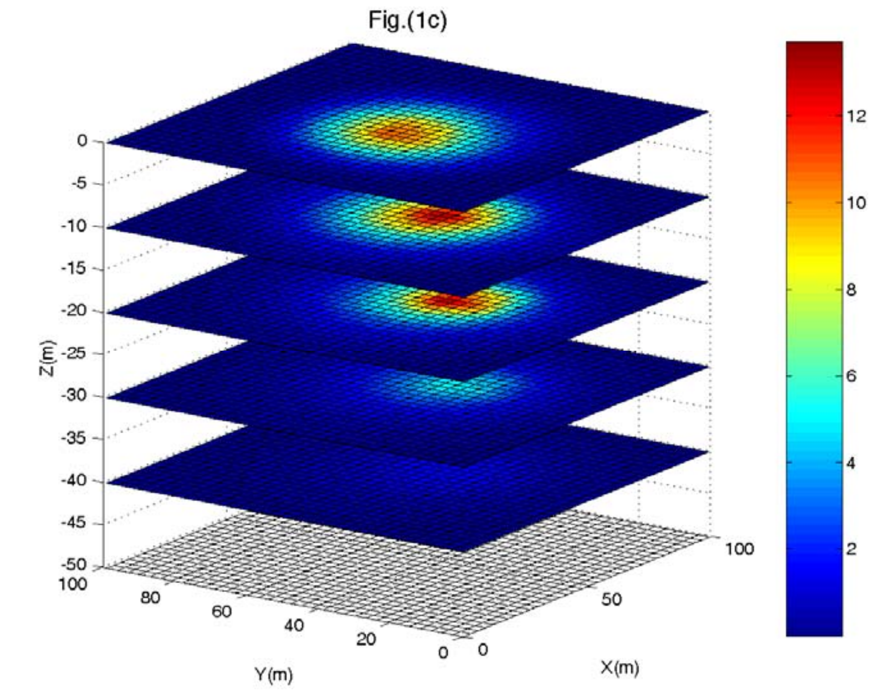

Figure 1(c). Inversion results by Guillen's method.

The results of inversion by using the Lewi's algorithm with and without considering the moment of inertia are reflected in Fig.(1b) and Fig.(1a) respectively. As it is clear the method by incorporating the moment of inertia (Fig. (1b)) shows better results.

The results of inversion applying the Guillen and Menichetti (1984) method are shown in

Fig.(1c) for comparison. Fig. (1c) does not show proper results which is due to the instabilities of Last and Kubik (1983) method (which is the base algorithm used in Guillen and Menichetti (1984) technique) in the case of contaminated data and three-dimensional gravity inversion reported by Lewi et al. (1994).

The second model is presented in Fig.(2). The data is also contaminated by noise. The results for Lewi' s method with and without considering the moment of inertia and Guillen's method are presented in Figs. (2a)-(2b) and (2c) respectively.

Fig. (2b) again shows the best results with the best depth resolution as it was expected. As this model is deeper than the model on Fig.(1) the minimization of inertia incorporating in Guillen's method partly compensate the instabilities of the Last and Kubik's algorithm and Fig.(2c) shows a relatively good results for this model.

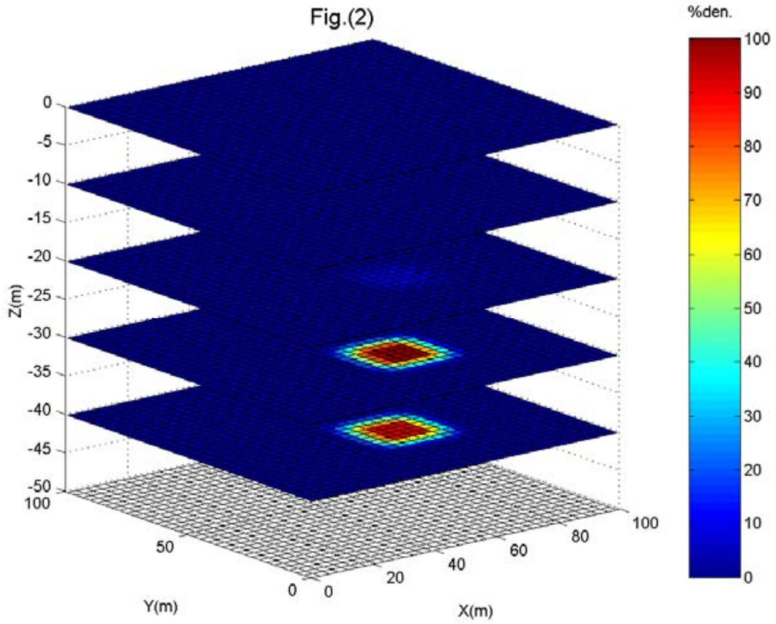

Figure 2. Synthetic model.

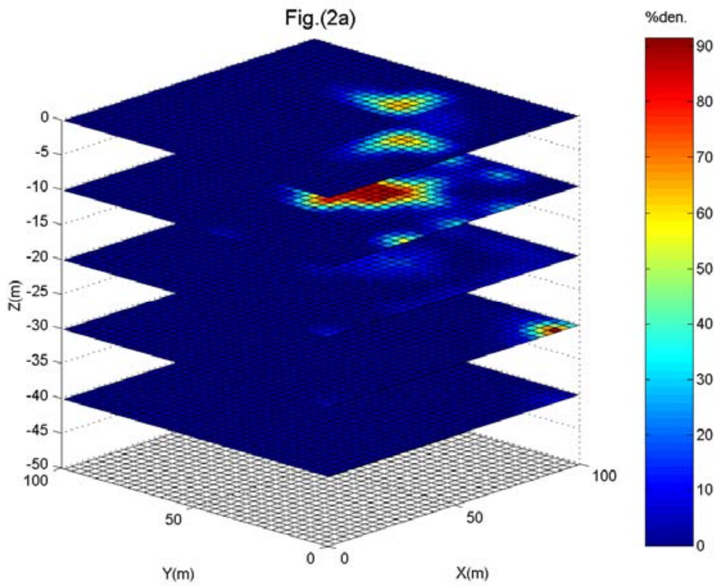

Figure 2(a). Inversion result by Lewi's algorithm.

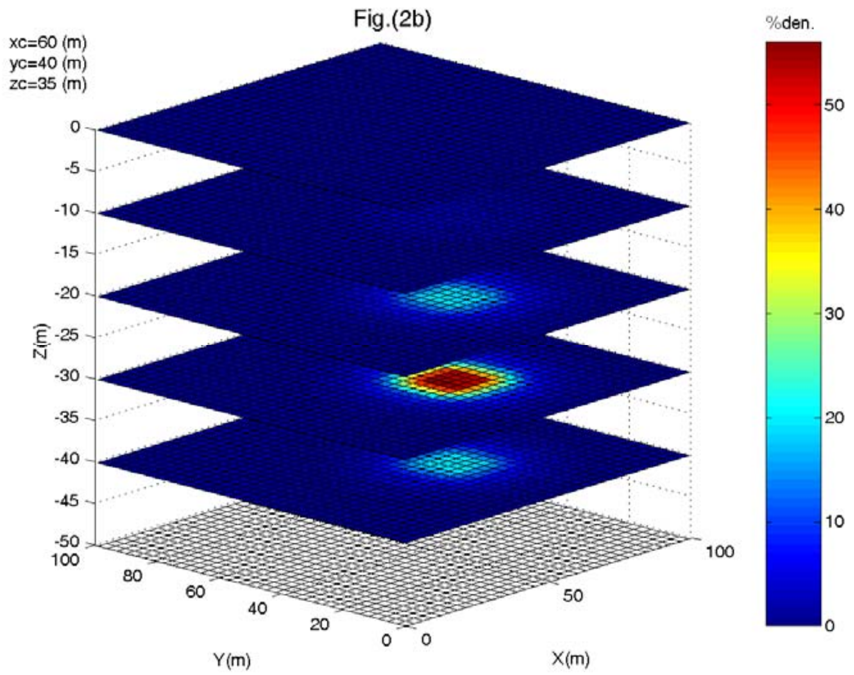

Figure 2(b). Inversion results by Lewi's algorithm including the minimum moment of inertia.

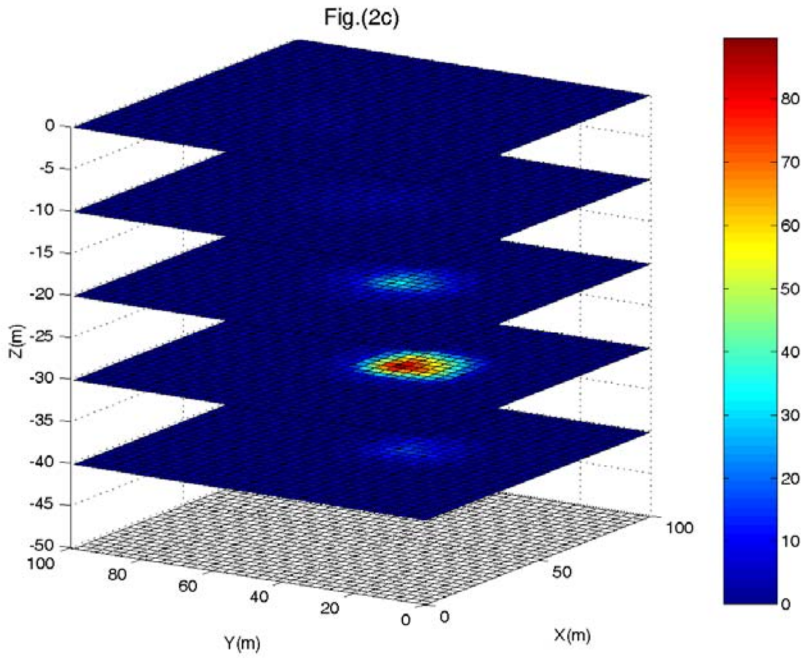

Figure 2(c). Inversion results by Guillen's method.

Final model is shown in Fig.(3). This model is more complex than the two previous ones. The gravity effect is again contaminated by noise. The results using the Lewi's method with and without the moment of inertia and Guillen's technique are presented in Figs. (3b)-(3a) and (3c) respectively. The best results belong to the Lewi's method with considering the moment of inertia (Fig.(3b)). As it has been expected the Guillen's method which is based on Last and Kubik algorithm is not stable when the source is complex

(Lewi et al., 1994). In addition to the contrast densities of the blocks demonstrated in the figures the RMS error can be another indication for comparing the results. The number of iteration and the RMS errors are shown in table (1).

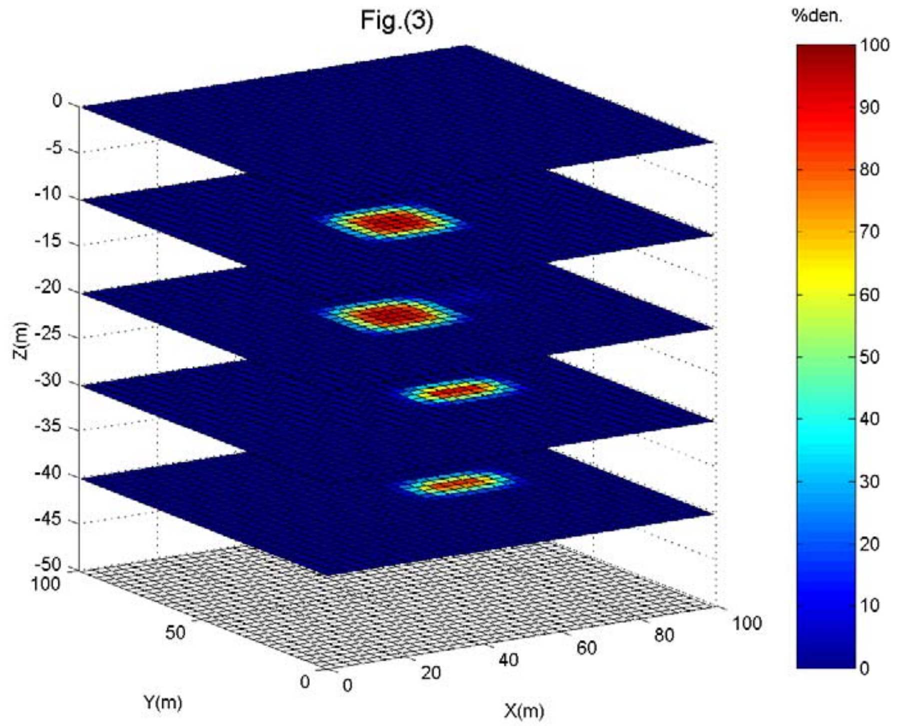

Figure 3. Synthetic model.

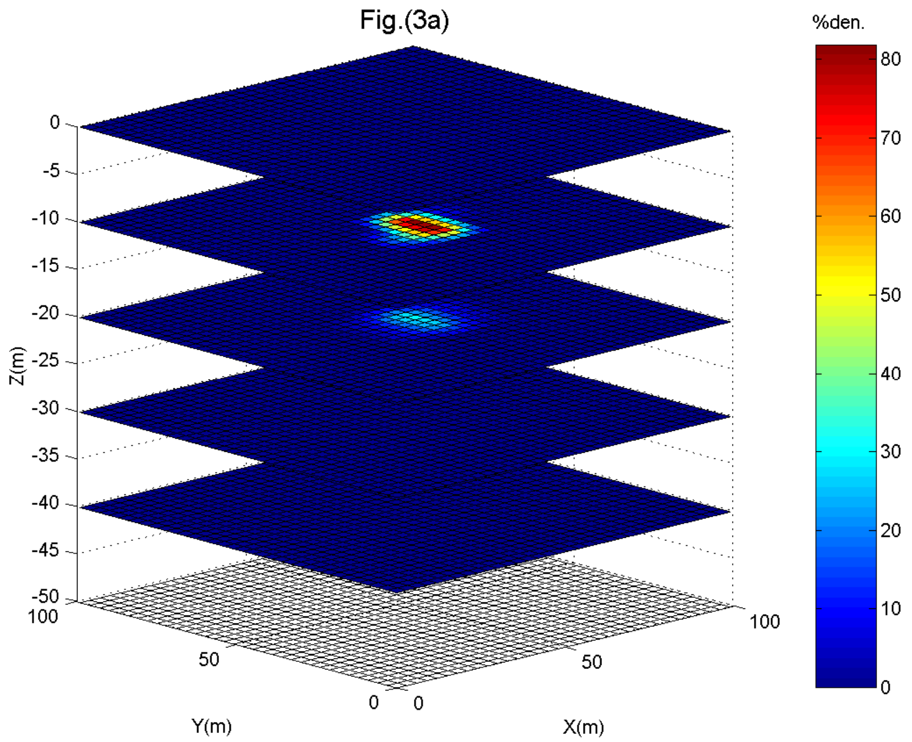

Figure 3(a). Inversion result by Lewi's algorithm.

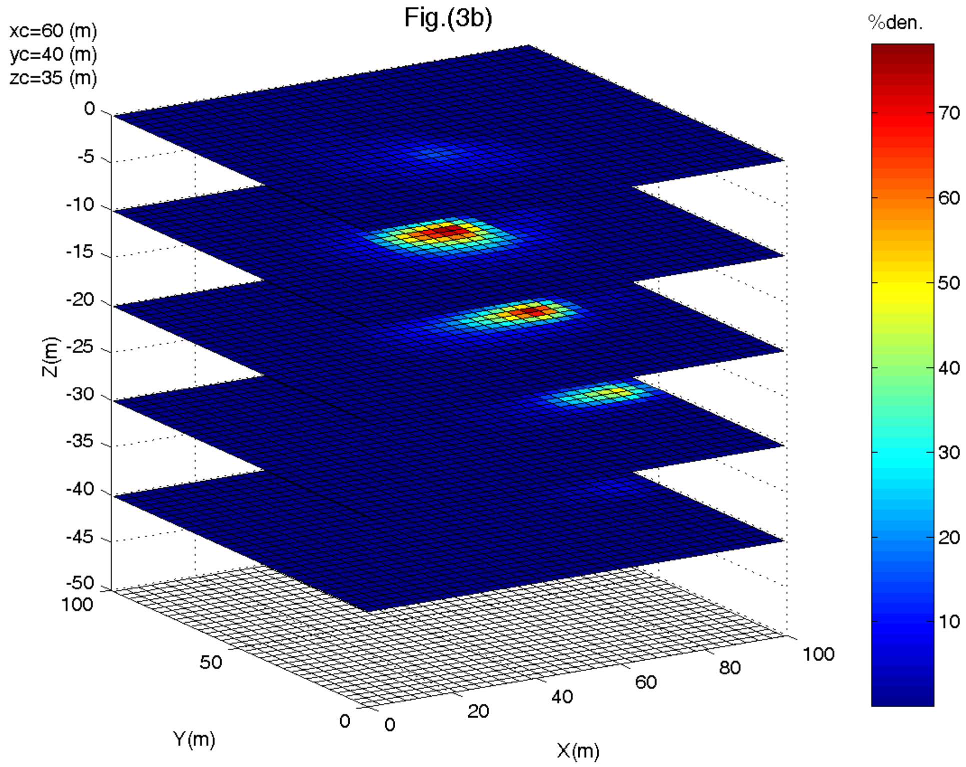

Figure 3(b). Inversion results by Lewi's algorithm including the minimum moment of Inertia.

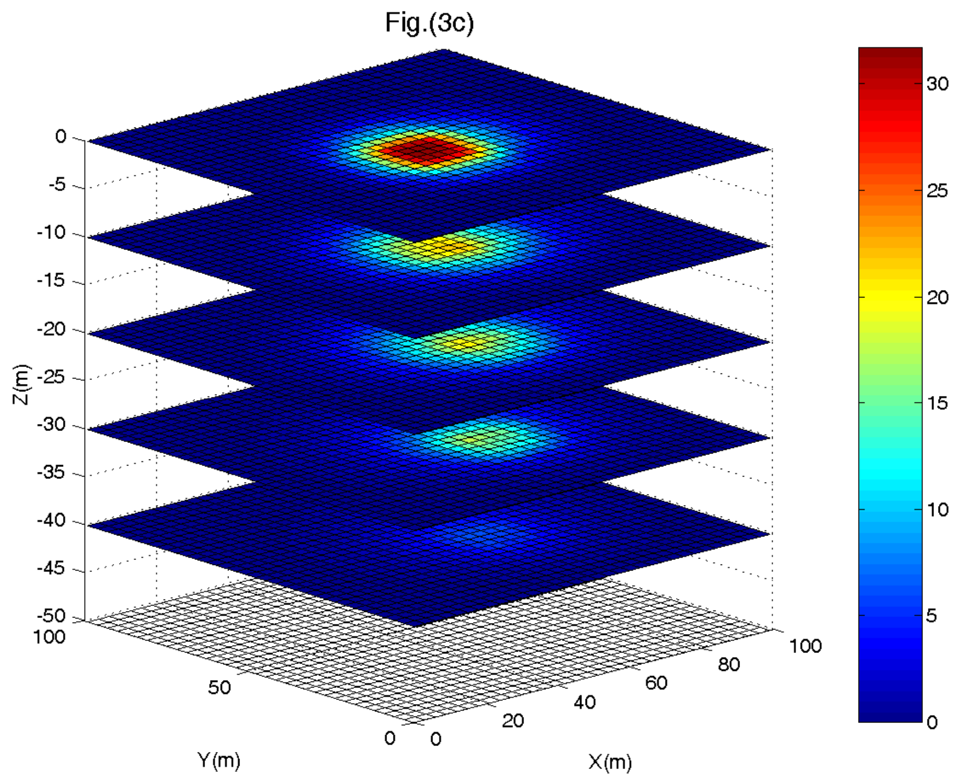

Figure 3(c). Inversion results by Guillen's method.

Table 1. Results of different methods.

Method	Fig.	xc(m)	yc(m)	zc(m)	Iteration	RMS (mGal)
Lewi	(1a)	60	40	25	4	2.92e-9
Lewi&M	(1b)	60	40	25	8	1.99e-9
Guillen	(1c)	60	40	25	12	2.5e-9
Lewi	(2a)	60	40	35	5	2.53e-6
Lewi&M	(2b)	60	40	35	18	9.45e-9
Guillen	(2c)	60	40	35	18	9.47e-9
Lewi	(3a)	60	40	35	4	1.03e-5
Lewi&M	(3b)	60	40	35	7	8.94e-6
Guillen	(3c)	60	40	35	7	9.54e-6

In this table "Lewi and M" refers to the Lewi's method with considering the minimum moment of inertia and xc, yc and zc are the coordinates of the center of the gravity when the minimum moment of inertia is included and are expected from a priori information which are quite vital in final results of the inversion.

In all these models the RMS error of "Lewi and M" is the smallest which is another indication of the advantage of this method.

Although the figures show similarity between Lewi and M and Guillen methods but the RMS error is mostly smaller in the first method. However in comparison of the methods the RMS error and the contrast densities demonstrated in the figures should be considered simultaneously.

7 FIELD EXAMPLE

The real data belong to the power plant area located close to Hamedan north-west of Iran where we were looking for the existence of sink holes.

The residual anomalies are presented in Fig.(4).

The results using the method are shown in Figs. (4a), (4b) and (4c).

Again the best results belong to Fig.(4b) applying Lewi's method and incorporating the moment of inertia. The depth of the anomaly is quite in agreement with the results obtaining through Euler depth which is demonstrated in Fig. (5).

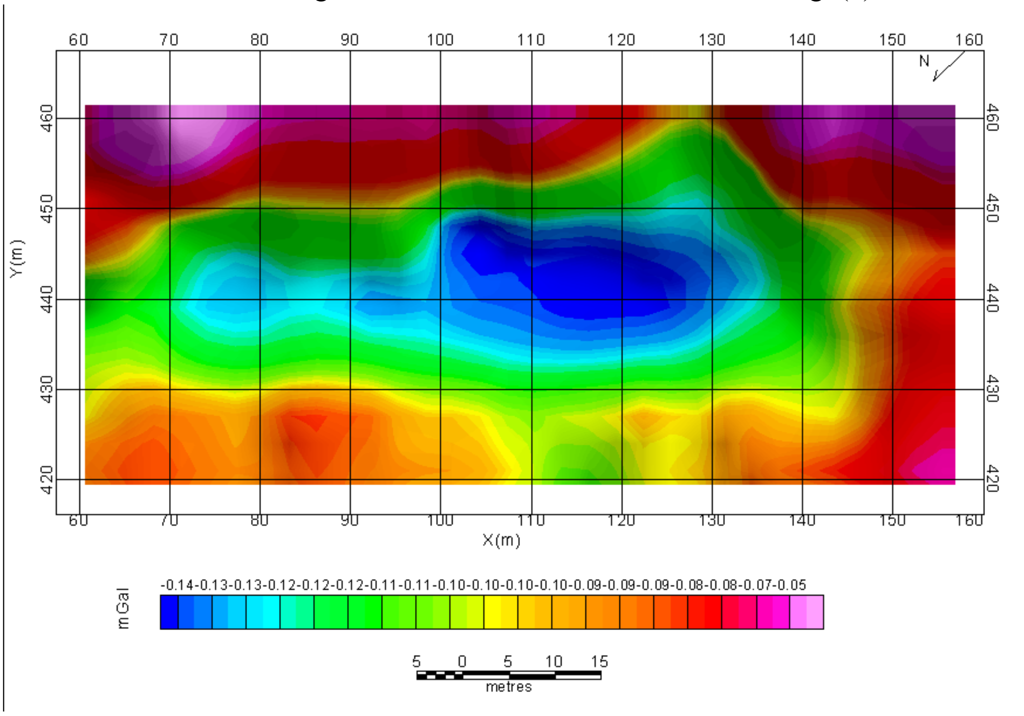

Figure 4. Residual anomalies of Camacho's example.

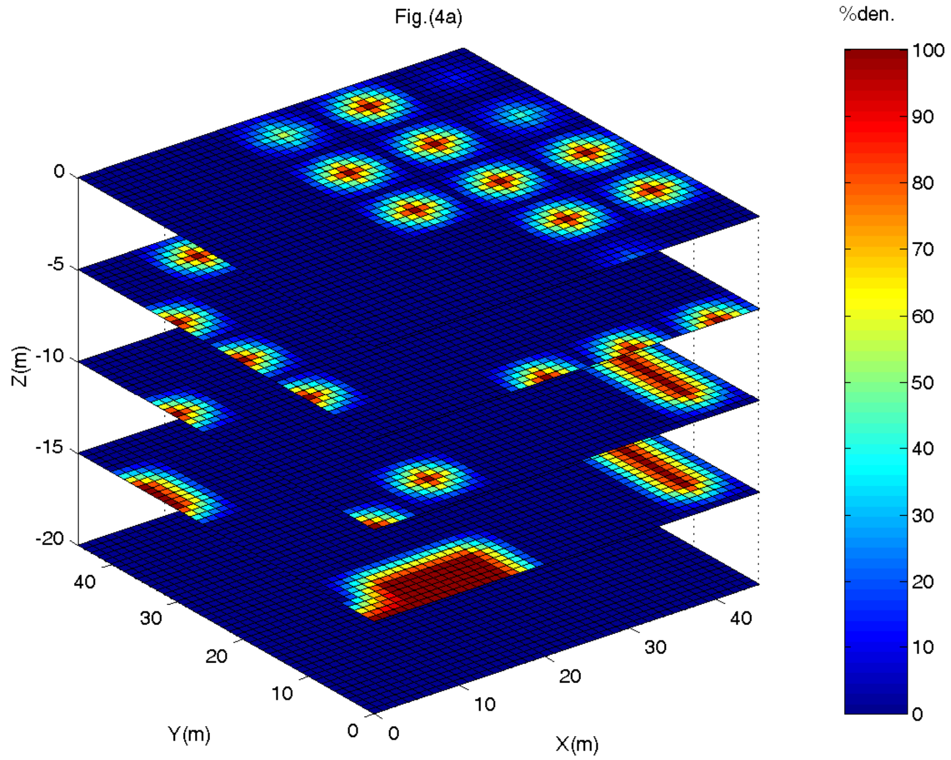

Figure 4(a). Inversion result by Lewi's algorithm.

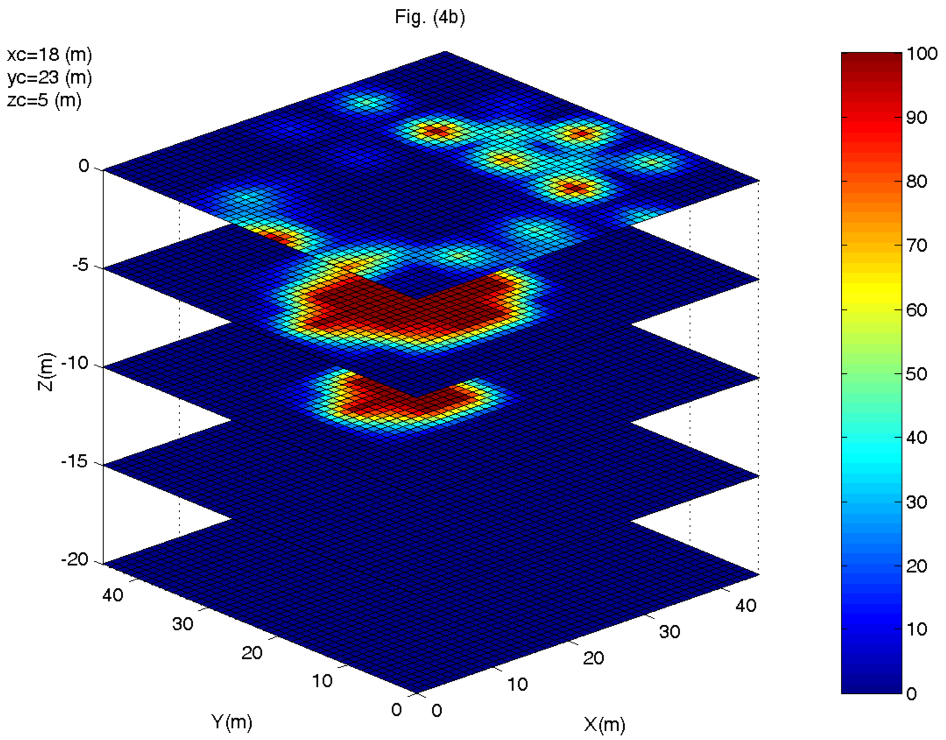

Figure 4(b). Inversion results by Lewi's algorithm including the minimum moment of inertia.

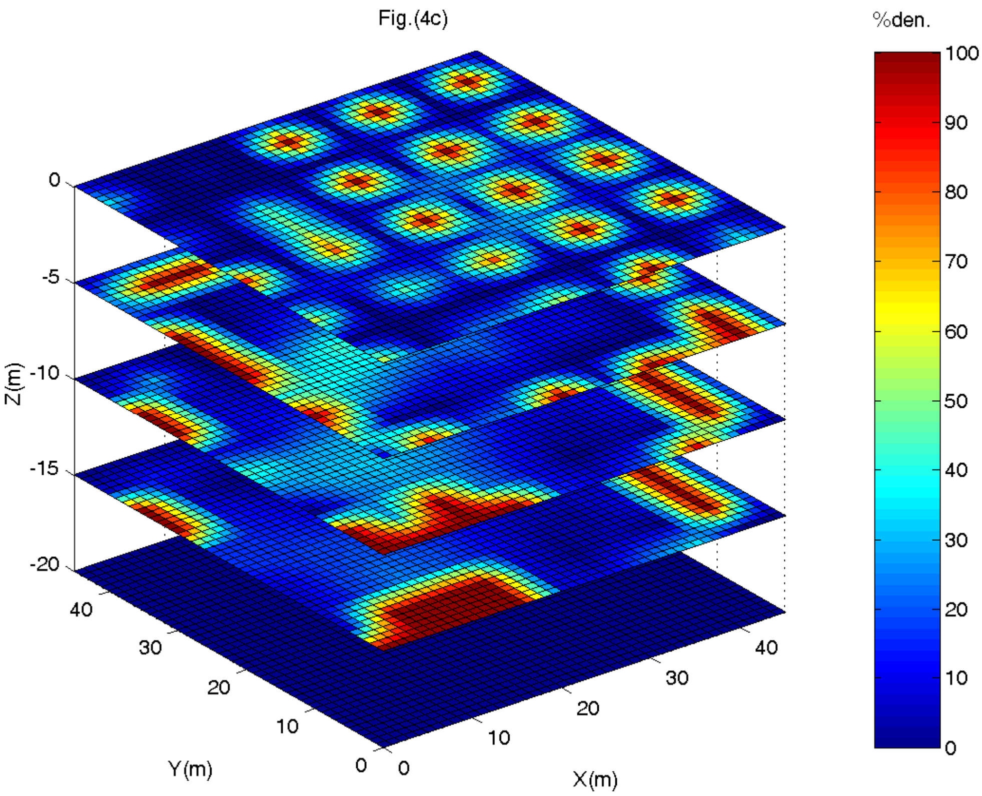

Figure 4(c). Inversion results by Guillen's method.

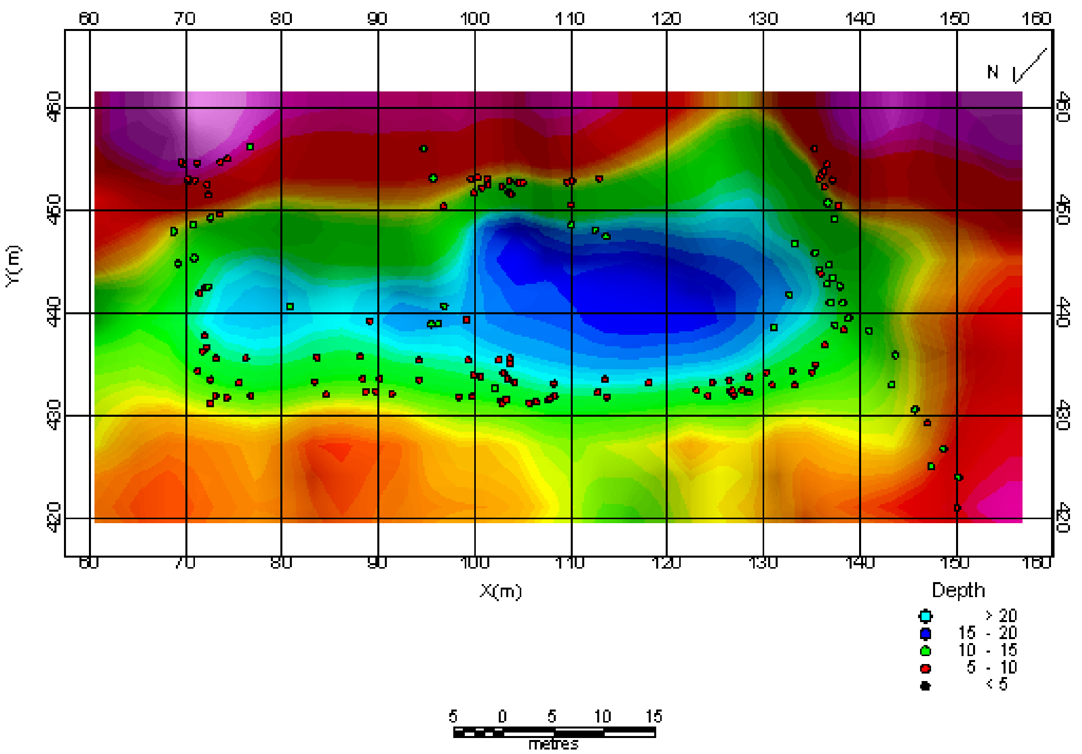

Figure 5. Euler depths.

8 CONCLUSION

The results of using compact gravity inversion (Last and Kubik, 1983) with minimizing the moment of inertia (Guillen and Menichetti, 1984) show some instabilities when we face to contaminated data.

In these cases the best results could be obtained when Lewi's algorithm (1997) including the minimizing of the moment of inertia is used.

Using this method the center of the gravity has to be determined with special care through a priori information.

ACKNOWLEDGEMENT

The authors are thankful to the authorities of Hamedan power plant and Moshanir and Toseh Olom Zamin consulting engineers companies and to the institute of Geophysics, Tehran university.

REFERENCES

Banerjee, B. and Gupta, S. P. D., 1977, Gravitational attraction of a rectangular parallel piped, Geophysics, 42(5), 1053-1055.

Barbosa, V. C. F. and Silva, J. B. C., 1994, Generalized compact gravity inversion, Geophysics, **59**, 57–68.

Guillen, A. and Menichetti, V., 1984, Gravity and magnetic inversion with minimization of a specific functional, Geophysics, 49(8), 1354-1360.

Ilk, k. H. 1993, Regularization for high resolution gravity field recovery by future satellite techniques. Inverse problems: Principles and applications in geophysics, technology, and medicine, in: Proceedings of the International Conference Held in Potsdam, athematical Research, **74**, Akademie Verlag.

Koch, K. R., 1988, Parameter estimation and hypothesis testing in linear models, Springer- Verlag, Berlin, Germany.

Last, B. J. and Kubik, K., 1983, Compact gravity inversion, Geophysics, 48(6), 713-721.

Lewi, E., 1997, Modeling and inversion of high-precision gravity data, Ph.D Thesis, Deutshe Geodaetische Kommission.

Lewi, E., Gupta, P. K. and Khattri, K. N., 1994, 2-D inversion of 3-D gravity anomalies, Proceedings of the national Academy of Science, India, 64(4A), 513-527.

Tarantola, A., 1987, Inverse problem Theory, Methods for data fitting and model parameter estimation, Elsevier, Amsterdam, The Netherlands.