Journal of the Earth & Space Physics. Vol. 33, No. 1, 2007, P. 13-18

Discrete wavelet transform of ellipsoidal Stokes integral for geoid determination

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(Received: 16 Nov 2005 , Accepted: 7 Feb 2007)

Abstract
This paper deals with the one-dimensional discrete wavelet transform (1D DWT) of four scaling coefficients are computed numerically by designing a convolutive operator.

The near-zone contribution of the integral is calculated through wavelet transform and for the far-zone contribution the classic expansion of the spherical harmonics applied.

Finally the geoidal heights are determined over a territory in Canada.

Key words: Ellipsoidal Stokes integral, Discrete wavelet transform, Geoidal heights

1 INTRODUCTION

The ellipsoidal Stokes integral as the solution of the ellipsoidal Stokes boundary-value problem was first defined by Martinec and Grafarend (1997). The direct numerical solution of the integral was expressed by Ardestani and Martinec (2000).

The direct numerical computation of the integral involves expose a relatively long and time consuming process considering the singularity of spherical and ellipsoidal Stokes functions.

Wavelet transform as a new tool for spectral solution of the integral could be quite fast and efficient.

Wavelet exhibits excellent localization properties that facilitate regional update and therefore the new observation can be used quickly up-date geoidal heights (Salamonowicz, 1999).

Moreover, we have quite a free hand to define the basis functions and corresponding coefficients in contrast to the Fourier transform which uses sine or cosine as the base functions.

2 DISCRETE WAVELET TRANSFORM

A wavelet is a small wave which has its energy concentrated in time or space to give a tool for analysis of transient, non-stationary or time, space-varying

$$f(x) = \sum_k \sum_j \alpha_{j,k} \psi_{j,k}(x) \tag{1}$$

where both j and k are integer indices and the $\psi(x)$ are the wavelet expansion that usually form an orthogonal basis. The two-dimensional parameterization is achieved from the function (mother wavelet),

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \qquad j,k \in Z, \tag{2}$$

Where Z is the set of all integers and $\psi \in L^2(R)$.

The mother wavelet can be generated from an associated function φ, the scaling function.

A function f is represented in such an orthonormal wavelet expansion by

$$f(x) = \sum_{j=-\infty}^{M} \sum_{k\in Z} \langle f, \psi_{j,k} \rangle \psi_{j,k}(x) + \sum_{k\in Z} \langle f, \ _{M,k} \rangle \ _{M,k}(x), \tag{3}$$

where indices j and k represent the dilation and translation parameters respectively. M is the maximum value of dilation. There are many different family of wavelets such as Daubechies, Coilfet etc.

3 CONVOLUTIVE OPERATOR

We aim to design the wavelet functions in such a way that we get a convolutive operator. We begin with continuous wavelet transform of f(x).

$$CWT = F(a,b) = \int_{-\infty}^{+\infty} f(x)\psi\left(\frac{x-a}{b}\right)dx. \tag{4}$$

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The inverse continuous wavelet transform will be

$$ICWT = f(x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(a,b)\psi\left(\frac{x-a}{b}\right)dadb. \quad (5)$$

The convolution of two functions g and h means,

$$F(x) = g * h = \int_{-\infty}^{+\infty} g(x-x')h(x')dx'. \quad (6)$$

Considering equation (5) yields,

$$g(x-x') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(a,b)\psi\left(\frac{x-x'-a}{b}\right)dadb. \quad (7)$$

Substituting equation (7) into eqn. (6) we have,

$$f(x) = \int_{-\infty}^{+\infty} h(x')\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(a,b)\psi\left(\frac{x-x'-a}{b}\right) dadb\right]dx' \quad (8)$$

where for a convolutive operator and satisfying the following equation,

$$F(a,b) = G(a,b)H(a,b) \quad (9)$$

the wavelet function ψ has to be defined in the form that, we are able to write,

$$\psi\left(\frac{x-x'-a}{b}\right) = \psi\left(\frac{x-a}{b}\right)\psi\left(\frac{x'-a}{b}\right). \quad (10)$$

Our approach to designing the convolutive operator is based on defining three known matrices G, D, N in such a way that the image of the elements of N in frequency space is equal to the multiplication of the image of the corresponding elements of G and D.

In other words, the problem is firstly assumed solved with a small value and then improved. The equations of the 2-D wavelet transforms (Burrus et al, 1998) are

$$f_j(n,m) = \sum_K\sum_L h(K-2n)h(L-2m) f_{j+1}(K,L), \quad (11)$$

$$d_{j+1}(n,m) = \sum_K\sum_L h_1(K-2n)h(l-2m) f_{j+1}(K,L), \quad (12)$$

$$d_{j+2}(n,m) = \sum_K\sum_L h(K-2n)h_1(L-2m) f_{j+1}(K,L), \quad (13)$$

$$d_{j+3}(n,m) = \sum_K\sum_L h_1(K-2n)h_1(L-2m) f_{j+1}(K,L). \quad (14)$$

where h and h_1 are down-sampling and up-sampling filters respectively. Supplying equation (11) and considering matrices D, G and N as a series, $f_{j+1}(K,L)$,

the following system of equations would be generated,

$$f_j(n,m) = g_j(n,m), \quad (15)$$

$$d_{j+1}f(n,m) = d_{j+1}g(n,m)d_{j+1}h(n,m), \quad (16)$$

$$d_{j+2}f(n,m) = d_{j+2}g(n,m)d_{j+2}h(n,m), \quad (17)$$

$$d_{j+3}f(n,m) = d_{j+3}g(n,m)d_{j+3}h(n,m). \quad (18)$$

Substituting m,n=0,1,2 we would have 15 independent equations with an unknown parameter (α) (Burrus et al, 1998).

By solving this non-linear system of equations numerically, the unknown parameter (α) and consequently the scaling coefficients would be determined as follows,

h0=0.01163397861446, h1=-0.01126316017171, h2=0.69547280257209, h3=0.71836994135826.

4 ELLIPSOIDAL STOKES INTEGRAL

The ellipsoidal Stokes integral (Martinec and Grafarend, 1997) is

$$N(b_0,\Omega)=\frac{b_0}{4\pi\gamma}\int\int_{\Omega'} f(\Omega')\left[S(\chi)-e_0^2 S^{ell}(\Omega,\Omega')\right]d\Omega', \qquad (19)$$

where x is the angular distance between directions Ω and Ω' , S(x) and are the spherical and ellipsoidal Stokes functions and, $S^{ell}(\Omega,\Omega)$ is the geoidal heights. $N(b_0,\Omega)$

Due to the lack of gravity anomaly f(Ω') on some parts of the globe, the integral is split into to the near-zone and the far-zone contributions,

$$N(b_0,\Omega)=N^{\chi_0}(b_0,\Omega)+N^{\pi-\chi_0}(b_0,\Omega), \qquad (20)$$

Is the near-zone contribution and $N^{x0}(b_0,\Omega)$ where is the far-zone contribution. $N^{\pi-x0}(b_0,\Omega)$.

5 NEAR-ZONE CONTRIBUTION

Computing the near-zone contribution of N, we have

$$N^{\chi_0}(b_0,\Omega)=\frac{b_0}{4\pi\gamma}\int_0^{\chi_0}\int_0^{2\pi} f(\Omega')\left[S(\chi)-e_0^2 S^{ell}(\Omega,\Omega'\right]d\Omega' \qquad (21)$$

implementing 1D DWT, we have,

$$N^{\chi_0}(b_0,\Omega)=\frac{b_0}{4\pi\gamma}\frac{\Delta\varphi\Delta\lambda}{4\pi}DWT_1^{-1}\{\sum_{\varphi_1}^{\varphi_n}DWT_1(f(\Omega'))DWT_1([S(\chi)-e_0^2 S^{ell}(\Omega,\Omega')]\sin\varphi)\}, \qquad (22)$$

where ΔΦ and Δλ are grid intervals in the directions of co-latitude and longitude

6 FAR-ZONE CONTRIBUTION

Computing the geoidal heights of far-zone contribution considering equation (19), we have

$$N^{\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{4\pi\gamma}\int_{\chi_0}^{\pi}\int_0^{2\pi} f(\Omega')\left[S(\chi)-e_0^2 S^{ell}(\Omega,\Omega')\right]\sin\chi d\chi d\alpha. \qquad (23)$$

This integral can be considered as a spherical Stokes integration extended by the term related to ellipsoidal contribution. We now split this integral as follows,

$$N^{\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{4\pi\gamma}\int_{\chi_0}^{\pi}\int_0^{2\pi} f(\Omega')S(\chi)\sin\chi d\chi d\alpha-\frac{b_0}{4\pi\gamma}\int_{\chi_0}^{\pi}\int_0^{2\pi} f(\Omega')e_0^2 S^{ell}(\Omega,\Omega')\sin\chi d\chi d\alpha. \qquad (24)$$

Since the magnitude of the second part is small, we approximate the far-zone contribution by just taking the first part of the right-hand side of equation (24) into account.

According to Heiskanen and Moritz (1967, equation (7-35)) we have,

$$N^{\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{2\gamma}\sum_{j=2}^{\infty}Q_j(\chi_0)\sum_{m=-j}^{j}f_{jm}Y_{jm}(\Omega),$$

where $N^{\pi-x0}(b_0,\Omega)$ are the geoidal heights of the far-zone contribution, $Q_j(x_0)$ are the Molodenkij truncation coefficients (Molodenskij, et al. 1960), f_{jm} could be determined by a global geopotential model (GGM, Heiskanen and Moritz, (1967)).

7 NUMERICAL RESULTS

Computing the near-zone contribution through equation (22) and by four computed scaling coefficients and the far-zone contribution by equation (25), the geoidal heights over an area in Canada (figure 1) are determined.

Instead of f (Ω'), we used Helmert gravity anomalies in the (5',5') grid model.

We also computed the geoidal heights by wavelet transform of the near-zone contribution through Daubechies' transform (Burrus et al, 1998) with wavelet coefficients as follows, and adding (table 1). to far-zone contribution equation (25), figure 2).

It is clear that are almost the same and quite smooth.

In the next part we computed the near-zone contribution through (Ardestani and Martinec, 2000),

$$N^{\chi_0}(b_0,\Omega)=\frac{b_0}{4\pi\gamma}\int_0^{\chi_0}\int_0^{2\pi}(f(\Omega')-f(\Omega))\left[S(\chi)-e_0^2S^{ell}(\Omega,\Omega')\right]d\Omega'+\frac{b_0}{4\pi\gamma}\int_0^{\chi_0}\int_0^{2\pi}\left[S(\chi)-e_0^2S^{ell}(\Omega,\Omega')\right]d\Omega',$$

and the far-zone contribution is computed by using equation (25) and the results are illustrated in figure 3.

Figure (3) clearly shows more details of geoidal heights than figures (1) and (2) which demonstrate the long-wavelength part of the geoidal heights.

8 CONCLUSIONS

Discrete wavelet transform is quite fast and if we apply it for near-zone contribution, the known coefficients such as Daub4 gives the same results as the computed coefficients of the convolutive operator.

However, the details of the geoid seems not to be declared through standard wavelets.

Therefore, considering the spherical wavelets (Freeden and Windheuser 1996) seems to be necessary and we hope to report on it in the near future.

Table 1. Daubechies N=4.

N	h (n)	h1(n)
0	0.48296291314453	0.12940952255126
1	0.83651630373781	0.22414386804201
2	0.22414386804201	-0.83651630373781
3	-0.12940952255126	0.48296291314453

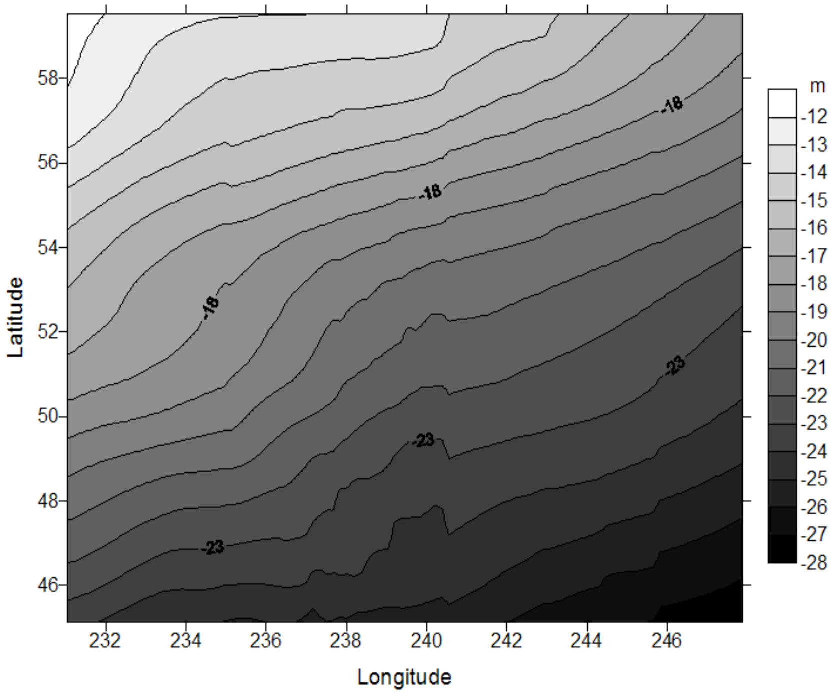

Figure 1. Geoidal heights (meter).

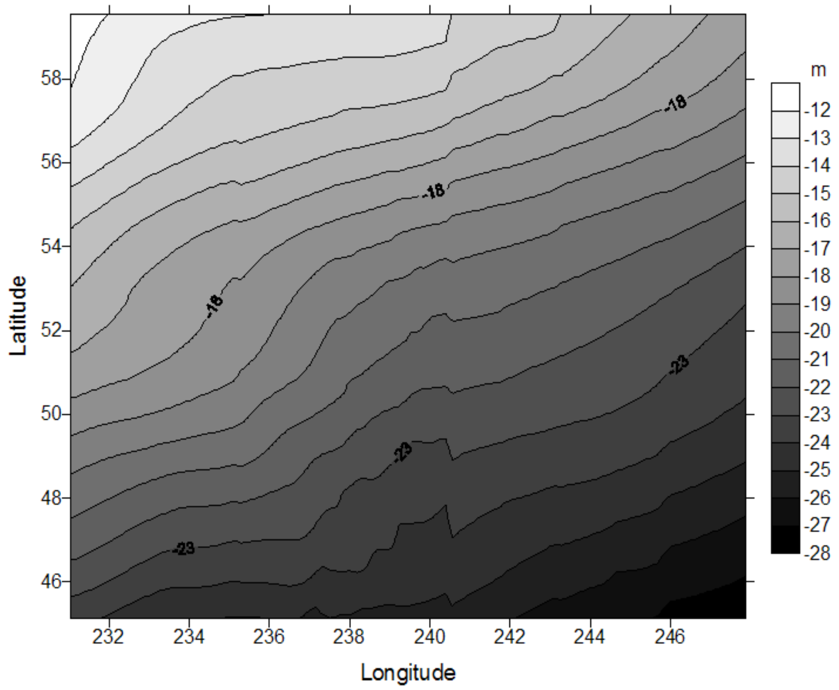

Figure 2. Geoidal heights (meter).

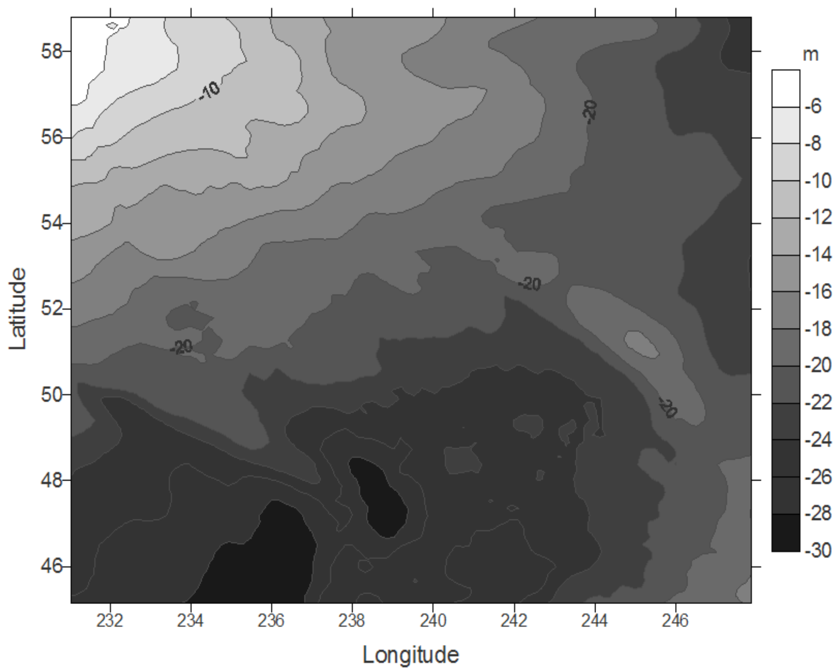

Figure 3. Geoidal heights (meter).

ACKNOWLEDGEMENTS

The author is thankful to the Institute of Geophysics, University of Tehran for all their support.

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