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1D FFT of ellipsoidal Stokes integral for geoid determination

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Abstract

One-dimensional fast Fourier transform (1D FFT) is used to solve the ellipsoidal Stokes integral (Martinec and Grafarend, 1997) in an ellipsoidal cap around the computational point (near-zone contribution) numerically.

For the far-zone contribution the spherical harmonic expansion can be applied. The geoidal height computation through direct numerical solution of the integral and 1D FFT will be compared for an area in Canada. The comparison shows relatively a great difference due to the application of FFT to the original ellipsoidal Stokes integral.

Keywords: Ellipsoidal Stokes integral, 1D FFT, Geoidal height

1 INTRODUCTION

The ellipsoidal Stokes integral as the solution of the ellipsoidal Stokes boundary-value problem has been defined by Martinec and Grafarend (1997). The direct numerical solution of the integral is expressed by Ardestani and Martinec (2000).

The determination of the geoid with spectral methods employing Stokes kernel function can perform with an FFT transform, due to the efficiency of the latter, and has been discussed by many authors (Sideris, 1997).

2 ELLIPSOIDAL STOKES BOUNDARY-VALUE PROBLEM

We introduce ellipsoidal coordinates (u, β, λ) through the transformation relations into Cartesian coordinates (x, y, z) (Heiskanen and Moritz, 1967),

$$\begin{aligned} x &= \sqrt{u^2 + E^2}\sin\beta\cos\lambda, \\ y &= \sqrt{u^2 + E^2}\sin\beta\sin\lambda, \\ z &= u\cos\beta, \end{aligned} \qquad (1)$$

where β is the reduced co-latitude, λ is the longitude, E is the linear eccentricity $E = \sqrt{a^2 - b^2}$ (=constant). The problem that we will deal with is to determine potential T(u,Ω), Ω=(β,λ), on and outside the reference ellipsoid of revolution $u = b_0$ so that:

$$\nabla^2 T = 0 \qquad \text{when} \qquad u > b_o, \qquad (2)$$

$$\frac{\partial T}{\partial u} + \frac{2}{u}T = -f \qquad \text{when} \qquad u = b_0, \qquad (3)$$

$$T \approx \frac{c}{u} + O(\frac{1}{u^3}) \qquad \text{when} \qquad u \to \infty \qquad (4)$$

where f(Ω) is assumed to be a known square integrable function, i.e., f(Ω)ε L_2(Ω), (gravity anomalies) and c is a constant and $O(\frac{1}{u^3})$ reflects the order of the error. According to Martinec and Grafarend (1997) the solution of the ellipsoidal Stokes boundary-value problem when $u = b_0$ is

$$N(b_0,\Omega) = \frac{\alpha_{00}(b_0)}{4\pi\gamma}\int_{\Omega_0} f(\Omega')d\Omega' + \frac{b_0}{4\pi\gamma} \int_{\Omega_0} f(\Omega')[S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')]d\Omega', \qquad (5)$$

where χ is the angular distance between directions Ω and Ω', and S(χ) is the spherical Stokes function (Heiskanen and Moritz, 1967) and Ω_0 is the full solid angle. According to Martinec and Grafarend (1997), $S^{ell}(\Omega,\Omega')$ is called the ellipsoidal Stokes function and has the same degree of singularity at point $\chi = 0$ as the spherical Stokes function. $\alpha =_{00}(b_0)$ has been defined by Martinec and Grafarend (1997),

$$\alpha =_{00} (u) = -\frac{Q_{00}(i\frac{u}{E})}{\frac{dQ_{00}(i\frac{u}{E})}{du}\Big|_{u-b_0} + \frac{2}{b_0}Q_{00}(i\frac{b_0}{E})} \tag{6}$$

where Q_{00} is the first degree of the second kind of Legendre's function. Removing the singularity is a crucial point in solving the integral numerically. In solving integral (5) numerically we resolve it into the near-zone contribution and the far-zone contribution,

$$N(b_0,\Omega) = N^{\chi_0}(b_0,\Omega) + N^{\pi-\chi_0}(b_0,\Omega), \tag{7}$$

where $N^{\chi_0}(b_0,\Omega)$ is the near-zone contribution and $N^{\pi-\chi_0}(b_0,\Omega)$ is the far-zone contribution.

3 NEAR-ZONE CONTRIBUTION

Computing the near-zone contribution of N in a cap $(C\chi_0)$ around the computational point, we have

$$N^{\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi\gamma}\int_{C\chi_0} f(\Omega')[S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')]d\Omega' \tag{8}$$

To abbreviate the near-zone integration domain we have used $C\chi_0$ which means $0 \le \chi \le \chi_0$ and $0 \le \alpha \le 2\pi$. Removing the singularity of χ=0 yields

$$N^{\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi\gamma}\int_{C\chi_0} (f(\Omega') - f(\Omega))[S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')]d\Omega' + \frac{b_0}{4\pi\gamma}\int_{C\chi_0} f(\Omega)[S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')]d\Omega'. \tag{9}$$

The direct numerical solution of this integral has been reported by Ardestani and Martinec (2000) as follows,

$$N^{\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi\gamma}\int_{C\chi_0} (f(\Omega') - f(\Omega))[S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')]d\Omega' - \frac{b_0}{2\gamma} f(\Omega)Q_0(\chi_0) - \frac{b_0}{4\pi\gamma}\int_{C\chi_0} f(\Omega)e_0^2 S^{ell}(\Omega,\Omega)d\Omega' \tag{10}$$

where $Q_0(\chi_0)$ is the zero-degree Molodenskij truncation coefficients(Molodenskij et al. 1960). The numerical solution of this equation is readily obtained by replacing the integrals by sums and substituting gravity anomalies instead of f (Ω) and f (Ω'). The analytic form of the third part of the right hand side is derived by Ardestani and Martinec (2000). A numerical evaluation of equation (8) for the near-zone contribution is given using the 1D FFT,

$$N^{\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi\gamma}\frac{\Delta\phi\Delta\lambda}{4\pi\gamma} F1^{-1}\left\{\sum_{\phi 1}^{\phi n} F1(f(\Omega'))F1([S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')]\sin\phi)\right\} \tag{11}$$

where F1 is the direct 1D FFT, $F1^{-1}$ is the inverse transform and φ, λ represent co-latitude and longitude. Since all quantities are real-valued, we can compute the Fourier transforms of the two real-valued arrays which are actually gravity anomalies simultaneously to save computer time (Huang et al. 2000)

4 FAR-ZONE CONTRIBUTION

Computing the geoidal heights of the far-zone contribution (outside the cap) considering equation (5), we find

$$N^{\pi-\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi\gamma}\int_{\chi=\chi_0}^{\pi}\int_{\alpha=0}^{2\pi} f(\Omega')[S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')]\sin\chi d\chi d\alpha. \tag{12}$$

This integral can be interpreted as a spherical Stokes integration extended by the term related to ellipsoidal contribution: $e_0^2\ S^{ell}(\Omega,\Omega')$; we now split this integral as follows,

$$N^{\pi-\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi\gamma}\int_{\chi=\chi_0}^{\pi}\int_{\alpha=0}^{2\pi} f(\Omega')S(\chi)\sin\chi d\chi d\alpha - \frac{b_0}{4\pi\gamma}\int_{\chi=\chi_0}^{\pi}\int_{\alpha=0}^{2\pi} f(\Omega')e_0^2 S^{ell}(\Omega,\Omega')\sin\chi d\chi d\alpha. \tag{13}$$

Since the magnitude of the second part is small, we approximate the far-zone contribution by taking just the first part of the right-hand side of equation (13) into account. The error of this approximation for the area in Canada is less than 5cm cm$ (Ardestani and Martinec 2003). According to Heiskanen and Moritz (1967).

$$N^{\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{2\gamma}\sum_{j=2}^{\infty}Q_j(\chi_0)\sum_{m=-j}^{j}f_{jm}Y_{jm}(\Omega), \tag{14}$$

where $N^{\pi-\chi_0}(b_0,\Omega)$ is the geoidal height of the far-zone contribution, $Q_j(\chi_0)$ are the Molodenskij truncation coefficients and f_{jm} can be determined by a global geopotential model GGM (Heiskanen and Moritz, 1967).

5 NUMERICAL RESULTS

Computing the near-zone contribution and the far-zone contributions through equations (10) and (14), we determined the geoidal heights over an area in Canada (figure 1). Instead of f(Ω'), we used Helmert gravity anomalies on a (5',5') grid. Applying equations (11) and (14), the geoidal heights are computed for the same area (figure 2). The absolute differences between these two methods are presented in figure 3. As can be seen through the figures 1 and 2, there is a relatively close correlation between the figures. Figure 3 indicates great differences (up to 9 meters) in geoidal heights computation.

6 CONCLUSIONS

Removing the singularity of the spherical and ellipsoidal Stokes functions in the near-zone contribution (equation 9) enables us to compute the geoidal heights precisely. Although the 1D FFT method is faster and does not need long computations and programming it introduces signficant errors in geoidal heights computations.

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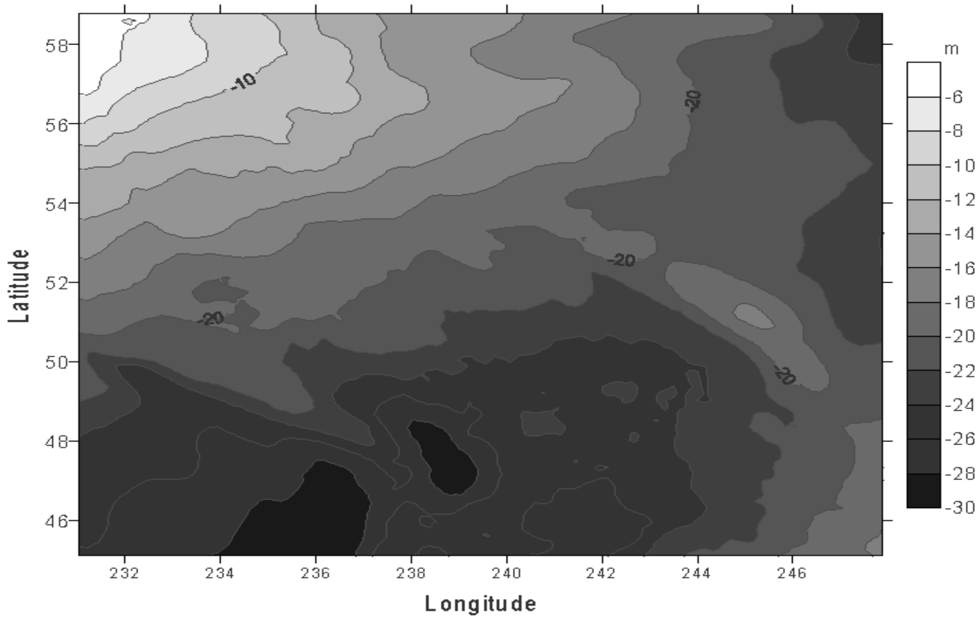

Figure 1. Geoidal heights through the direct numerical solution equations 10 and 14 (in meters).

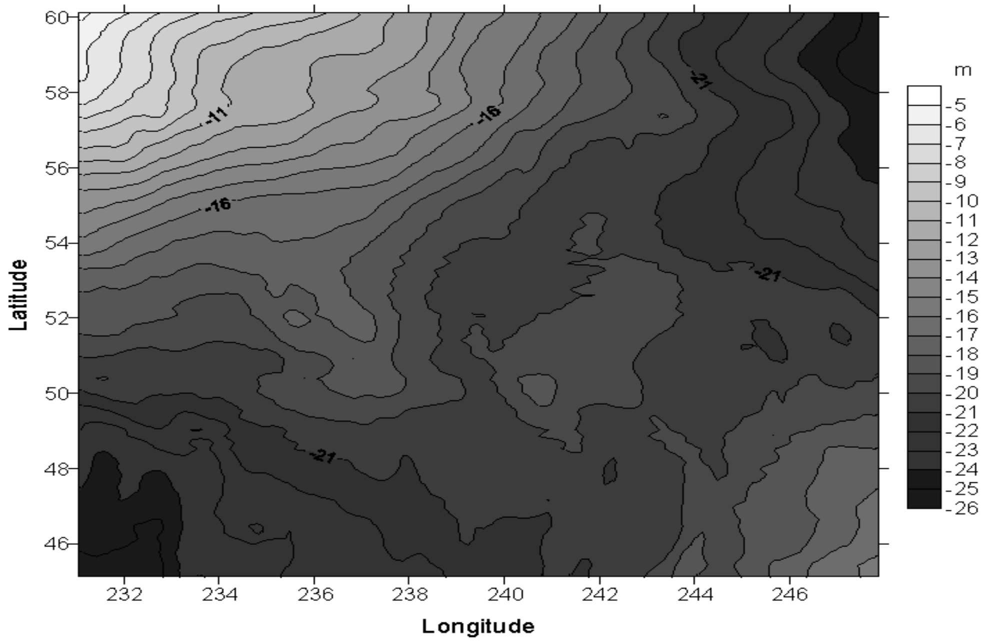

Figure 2. Geoidal heights through 1D FFT equations 11 and 14 (in meters).

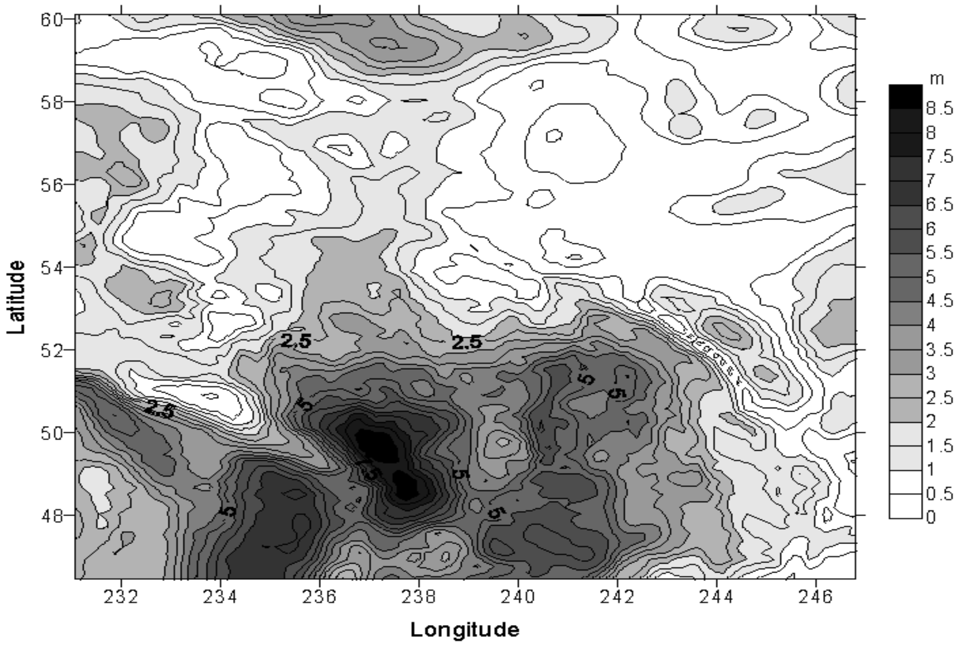

Figure 3. The differences between the geoidal heights computation through the direct numerical solution equations 10 and 14 and 1D FFT equations 11 and 14 (in meters).