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The Iranian geoid by high-degree ellipsoidal Stokes integral

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Abstract

The high-degree ellipsoidal Stokes integral (Ardestani and Martinec, 2003a) is used to compute the geoidal heights over a territory in Iran. To compute the low-degree part of the geoidal heights a global geopotential model (EGM96) is used and for the high-degree part, the solution of the ellipsoidal Stokes boundary-value problem (BVP) in the form of surface integral is applied.

Therefore the geoidal heights con be calculated for a part of Iranian territory where the data is available.

Keywords: Iranian geoid, High-degree ellipsoidal Stokes integral

1 INTRODUCTION

Martinec and Grafarend (1997) formulated the ellipsoidal Stokes BVP and found its solution in a closed form as an integral over the Earth's surface. However, due to the lack of global coverage of terrestrial gravity survey, the surface integral has to be truncated to the near-zone and far-zone in practical applications.

The near-zone contribution has been treated by Ardestani and Martinec (2000). The far-zone contribution is computed by the method introduced by Ardestani and Martinec (2003b).

The truncation of the integral may cause a serious loss of long-wavelength information on the geoid. This was a crucial problem before the satellite epoch.

Nowadays, we involve the information on gravitational field coming from satellite observations into the geoid determination in order to recover the last parts of information.

2 ELLIPSOIDAL STOKES BOUNDARY-VALUE PROBLEM

We consider the ellipsoidal coordinates {u,β,λ} (Heiskanen and Moritz, 1967), where β is the reduced co-latitude, λ is the longitude and E is the linear eccentricity,

$$E=\sqrt{a^2+b^2}.$$

The ellipsoidal Stokes BVP can be formulated as follows. Determine the disturbing potential T(u,Ω), Ω=(β,λ) on and outside the reference ellipsoid of revolution u=b so that:

$$\nabla^2 T = 0 \quad \text{if} \quad u > b_0 \tag{1}$$

$$\frac{\partial T}{\partial u} + \frac{2}{u} T = -f \quad \text{if} \quad u = b_0 \tag{2}$$

$$T \approx \frac{c}{u} + O\left(\frac{1}{u^3}\right) \quad \text{if} \quad u \to \infty \tag{3}$$

where f(Ω) is assumed to be a known square integrable function (gravity anomalies) and c is a constant. The spectral form of the solution to this BVP has been constructed by Martinec and Grafarend (1997):

$$T(u,\Omega) = \sum_j^{\infty} \sum_m f_{jm} \alpha_{jm}(u) Y_{jm}(\Omega). \tag{4}$$

where T is the disturbing potential, f_{jm} are the expansion coefficients of f(Ω) to be determined from boundary condition (2) and α(u) has been defined by Martinec and Grafarend (1997, equation 13),

$$\alpha_{jm}(u) = \frac{Q_{jm}\left(i\frac{u}{E}\right)}{\dfrac{dQ_{jm}\left(i\frac{u}{E}\right)}{du}\Big|_{u=b_0} + \dfrac{2}{b_0} Q_{jm}\left(i\frac{b_0}{E}\right)} \tag{5}$$

The solution of the problem at the point $u = bo$ can be expressed in the integral form (ibid., equation 47):

$$T(b_0,\Omega) = \frac{\alpha_{00}(b_0)}{4\pi}\int_{\Omega_0} f(\Omega')d\Omega' + \frac{b_0}{4\pi} \int_{\Omega_0} f(\Omega')[S(\chi) - e_0^2 S^{ell}(\Omega,\Omega')d\Omega' \quad (6)$$

where Ω is the full solid angle, χ is the angular distance between the directions Ω and Ω', $S(\chi)$ is the spherical Stokes function (Heiskanen and Moritz, 1967) and $S^{ell}(\Omega,\Omega')$ is the ellipsoidal Stokes function(Martinec and Grafarend, 1997). Note that function $S^{ell}(\Omega,\Omega')$ has the same degree of singularity at the point as does function $S(\chi)$. We now split the disturbing potential T into a low-degree reference potential $T_l^{ell}(b_0,\Omega)$ and a higher-degree potential $T^{ell,l}(b_0,\Omega)$. Likewise, the gravity anomaly $f(\Omega)$ is split into a low-degree part $f_l(\Omega)$ and a high-degree part $f^l(\Omega)$. The low-degree part of the disturbing potential will be represented in the spectral form,

$$T_l(b_0,\Omega) = \sum_j^l \sum_m f_{jm}\alpha_{jm(b_0)Y_{jm}(\Omega)} \quad (7)$$

which results from equation (4) for $u = bo$. The upper limit l in the summation over angular degree j represents the cut of degree of a reference potential. To compute the high-degree part $T^{ell}(b_0,\Omega)$, we use the integral representation (6).

We split the spherical and the ellipsoidal Stokes functions to the low-degree and high-degree parts and use the orthogonally property of the spherical harmonics (Heiskanen and Moritz, 1967), and we obtain

$$T^l(b_0,\Omega) = \frac{b_0}{4\pi}\int_{\Omega_0} f^l(\Omega')[S^l(\chi) - e_0^2 S^{ell,l} (\Omega,\Omega')d\Omega' \quad (8)$$

where $S^l(\chi)$ and $S^{ell,l}(\chi)$ are the high-degree parts of the spherical and ellipsoidal Stokes functions, respectively. Because of numerical computations, it is convenient to decompose the integral(8) into the near-zone contribution $T^{l,\chi_0}(b_0,\Omega)$ and the far-zone contribution $T^{l,\pi-\chi_0}(b_0,\Omega)$ that is,

$$T^l(b_0,\Omega) = T^{l,\chi_0}(b_0,\Omega) + T^{l,\pi-\chi_0}. \quad (9)$$

3 NEAR-ZONE CONTRIBUTION

The near-zone contribution of the high-degree part of the disturbing potential is determined by the integral in equation (8) but with the integration domain shrank to the integration cap $C\chi_0$ of radius χ_0 that surrounds the computation point χ=0:

$$T^{l,\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi}\int_{C\chi_0} f^l(\Omega')[S^l(\chi) - e_0^2 S^{ell,l} (\Omega,\Omega')]d\Omega' \quad (10)$$

To remove the singularity of the Stokes functions at the point χ=0, we write

$$T^{l,\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi}\int_{C\chi_0} [f^l(\Omega') - f^l(\Omega)][S^l(\chi) - e_0^2 S^{ell,l}(\Omega,\Omega')]d\Omega' - \frac{b_0}{4\pi}\int_{C\chi_0} f^l(\Omega')[S^l(\chi) - e_0^2 S^{ell,l}(\Omega,\Omega')]d\Omega' \quad (11)$$

and finally we have (Ardestani and Martinec 2003a)

$$T^{l,\chi_0}(b_0,\Omega) = \frac{b_0}{4\pi}\int_{C\chi_0} [f^l(\Omega') - f^l(\Omega)][S^l(\chi) - e_0^2 S^{ell,l}(\Omega,\Omega')]d\Omega' \frac{b_0}{2} f^l(\Omega)\left[-Q_0(\chi_0) + \sum_{j=2}^{l} \frac{2j-1}{j-1} R_{j0}(\cos\chi_0)\right] - \frac{b_0}{4\pi} f^l(\Omega) e_0^2 \quad (12)$$

$$\int_{C\chi_0}\left\{S^{ell}(\Omega-\Omega') - 4\pi\sum_{j=2}^{l}\sum_m \frac{1}{(j-1)^2}\left[\frac{(j+1)^2 - m^2}{2j+3} - (j+1)Y_{jm}(\Omega)Y^*_{jm}(\Omega')\right]\right\}d\Omega'$$

4 FAR-ZONE CONTRIBUTION

The far-zone contribution to integral (8) has the following form

$$T^{1,\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{4\pi}\int_{\alpha=0}^{2\pi}\int_{\chi_0}^{\pi} f^1(\Omega')\left[S^1(\chi)-e_0^2 S^{ell,1}(\Omega,\Omega')\right]\sin\chi \; d\varphi \; d\alpha \quad (13)$$

Following the same strategy introduced by Ardestani and Martinec (2003b), the integral can be considered as a spherical Stokes integration extended by the term related to ellipsoidal contribution $e_0^2 S^{ell}(\Omega,\Omega')$; we now split this integral as follows,

$$T^{1,\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{4\pi}\int_{\alpha=0}^{2\pi}\int_{\chi_0}^{\pi} f^1(\Omega')\left[S^1(\chi)\right]\sin\chi \; d\varphi \; d\alpha-\frac{b_0}{4\pi}\int_{\alpha=0}^{2\pi}\int_{\chi_0}^{2\pi} f^1(\Omega')\left[e_0^2 S^{ell,1}(\Omega,\Omega')\right]\sin\chi \; d\varphi \; d\alpha \quad (14)$$

Taking the expression defined by Heiskanen and Moritz (1967) into account for the first part of the right-hand side of equation (14) we obtain

$$T^{1,\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{2}\sum_{j=l+1}^{\infty}Q_j^1(\chi_0)\sum_m f_{jm}Y_{jm}(\Omega)-\frac{b_0}{4\pi}-\frac{b_0}{4\pi}\int_{\alpha=0}^{2\pi}\int_{\chi_0}^{2\pi} f^1(\Omega')\left[e_0^2 S^{ell,1}(\Omega,\Omega')\right]\sin\chi \; d\varphi \; d\alpha \quad (15)$$

where $T^{1,\pi-\chi_0}(b_0,\Omega)$ is the disturbing potential of the far-zone contribution, $Q_j^1(\chi_0)$ are the Molodenskij truncation coefficients(Molodensij et al. 1960) and f_{jm} can be determined by a global geopotential model (GGM) (Heiskanen and Moritz, 1967).

To compute the second part of the right-hand side of equation (15), we introduce the new function $\tilde{S}^{ell,1}(\Omega,\Omega')$ as follows,

$$\tilde{S}^{ell,1}(\Omega,\Omega')=\begin{cases}0 \quad \text{if} \quad 0<\chi<\chi_0\\ S^{ell,1}(\Omega,\Omega') \quad \text{if} \quad \chi_0<\chi<\pi\end{cases}$$

Expanding function $\tilde{S}^{ell,1}(\Omega,\Omega')$ in a series of spherical harmonics we obtain

$$\tilde{S}^{ell,1}(\Omega,\Omega')=\sum_{j=l+1}^{\infty}\sum_m q_{jm}^1 Y_{jm}(\Omega)Y_{jm}^*(\Omega'). \quad (16)$$

Multiplying two sides of equation (16) in

$$\iint_{\Omega'} Y_{j1m1}^*(\Omega')d\Omega'$$

where Ω' is the full solid angle and keeping in mind the orthogonally property of the spherical harmonics we have,

$$\iint_{\Omega'}\tilde{S}^{ell,1}(\Omega,\Omega')Y_{jm}^*(\Omega')d\Omega'=\sum_{j=0}^{\infty}\sum_m q_{jm}^1 Y_{jm}(\Omega). \quad (17)$$

Now by substituting equation (17) into the second part of the right-hand side of equation (15) and expanding the $f(\Omega)$ in a series of spherical harmonics and taking into account the orthogonally property of the spherical harmonics we finally obtain,

$$T^{\pi-\chi_0}(b_0,\Omega)=\frac{b_0}{2}\sum_{j=l+1}^{\infty}Q_j^1(\chi_0)\sum_m f_{jm}Y_{jm}(\Omega)-e_0^2\frac{b_0}{4\pi}\sum_{j=l+1}^{\infty}\sum_m f_{jm}q_{jm}^1 Y_{jm}(\Omega). \quad (18)$$

where $q_{jm}Y_{jm}(\Omega)$ can be computed through equation (17) by solving the left-hand side of equation numerically and then solving the system of linear equations.

5 NUMERICAL RESULTS

The first step in geoid computation is providing the necessary data, preferably Helmert gravity anomalies. These anomalies can be computed from free-air anomalies including the topography corrections.

These necessary data have been obtained from the Iranian National Cartography Center (NCC) for more than 6000 points distributed over Iranian territory. The data have been arranged in a grid (2.5', 2.5') suitable for use as the input to computer code.

Meanwhile the gap between the data are filled by interpolation. The other set of data that can be used in the code is geopotential gravity models such as EGM96 or EGM2000 which are downloaded from the internet.

Using EGM96 for f_{jm} in equation (7) and Bruns formula,

$$N = \frac{T}{\gamma} \tag{19}$$

the low-degree part of geoidal heights are computed. Then the near-zone contribution of geoidal heights are obtained through equation (12) by considering a cap around the computational point by the radius equal to 6 degree.

For using this equation (12) the parameters such as zero-degree Molodenskej (Q_0) and Paul coefficients (R_{j0}) are computed in different subroutines in computer code. The Helmert gravity anomalies are substituted instead of ($f^1(\Omega)$, $f^1(\Omega')$).

For the far-zone contribution equation (18) and equation (19) are used. In equation (18) the new coefficients q^l_{jm} are obtained by solving the linear system of equation (17).

Adding the low-degree to the high-degree part, the near-zone and the far-zone contributions the geoidal heights are computed and shown in figure 1. By using the digital terrain model of Iran the topographical curves are also demonstrated in figure 2. There is a good spatial correlation between figure 1 and the field figure 2.

The accuracy of the program can be tested by the method expressed in Ardestani and Martinec (2000).

6 CONCLUSIONS

The geoidal heights are computed through high-degree ellipsoidal Stokes integral efficiently. The close correlation between topographical and geoidal heights show the correctness of the computations implicitly. However, by expanding the data file covering the whole territory of Iran and some neighboring countries the geoidal heights could be obtained more accurately.

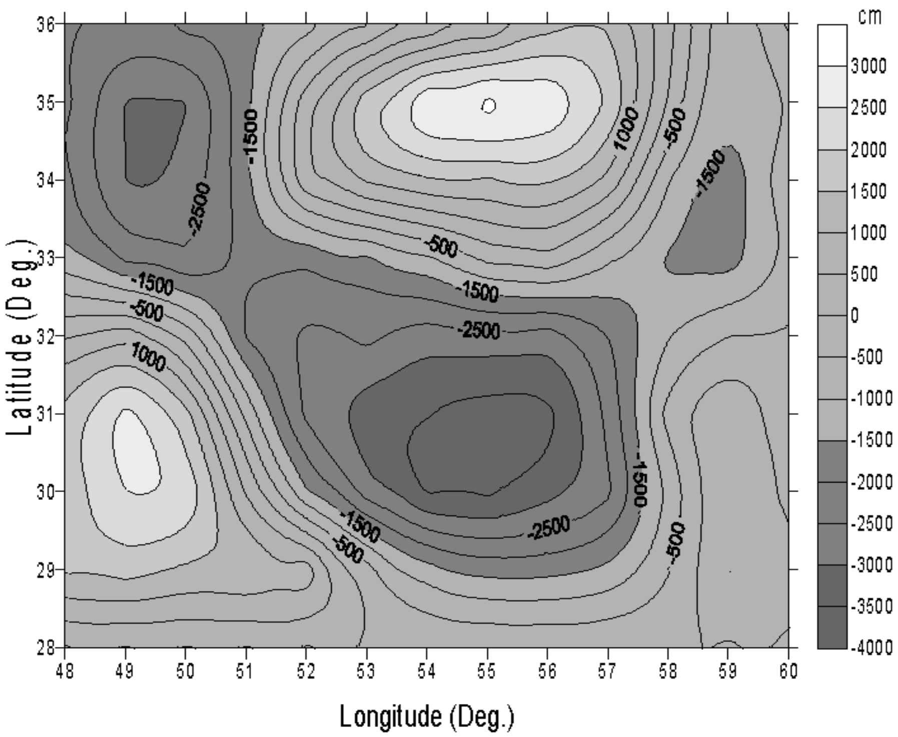

Figure 1. The geoidal heights (cm).

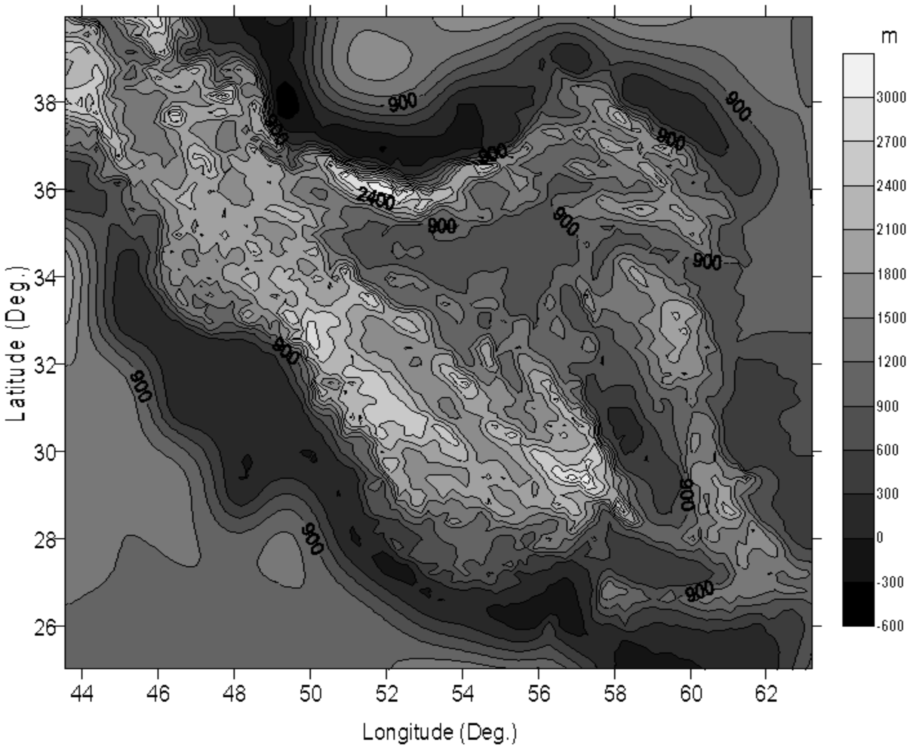

Figure 2. The topographical heights (m).

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